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by

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ABSTRACT

A new class of non-decreasing stochastic processes is characterized. These processes satisfy a generalization of the notion of an increasing failure rate. From physical considerations, these processes seem suitable for describing the process of cumulative wear or damage. The main interest with the model is an investigation of the first time until the process exceeds a random barrier.

For this class of processes, it is shown that the first passage time random variable across a random barrier has an increasing failure rate, regardless of the distribution of the barrier. In addition, by the use of certain intuitive, non-parametric assumptions, tight bounds on the moments of this first passage time random variable are obtained.

INTRODUCTION

Few mechanistic models have been developed for processes which result in the failure of components. This is largely due to the great number of difficulties involved in trying to infer something about the underlying physical mechanism of wear from observations of the failure times.

H.E. Daniels (1945) and Z.W. Birnbaum and S.C. Saunders (1958) have proposed mathematical models justifying the assumption of the normal and gamma families respectively, for the distribution of the failure times in certain situations. A. Mercer (1961) considered the case in which the wear process consisted of blows occurring according to a Poisson process, the amount of wear per blow having a gamma distribution. He assumes further that the probability that the component fails in the interval $(t, t + dt)$, given the wear is x, is of the form $\lambda_1(t) + \lambda_2 x$. He obtains reasonably simple results when the amount of wear per blow is constant or an exponential random variable. E. Parzen (1959) considers the cumulative damage to a component subject to a large number of blows at a finite number of stress levels.

The model of wear we shall consider is one in which a component wears out according to a non-decreasing stochastic process $\{Z_t \mid t > 0\}$, where Z_t, a random variable, represents the cumulative damage at time t. Now, rather than focusing our main attention on certain selected wear processes governed by mathematically convenient families of distributions, we shall take the perhaps more realistic approach of studying certain non-parametric classes of processes based on physical considerations. We shall further assume that the component functions until the cumulative wear exceeds some critical threshold, in general, a random variable, at which time it breaks or otherwise becomes useless. It is this first passage time until breakdown which is of special interest in this investigation.

The wear of railroad tracks appears to be well described by this sort of model. One reason for replacing the track is that the weight per foot of the track decreases after use, and the track is considered too dangerous to be used if the weight per foot drops below some predetermined level. The wear in the track is due to a number of causes. When a train goes around the curve of a track, the flanges of the wheel repeatedly hit the rails. The force and frequency of these blows depend on the weight and speed of the train. A second cause of wear is that adjacent rails may have settled to slightly different levels, so that the train drops from one level to the next. A third and less destructive type of wear is the nearly continuous abrasive action due to nature. These three wear mechanisms will be of different degrees of importance in different sections of the track.

Summary:

In this paper, we characterize a new class of non-decreasing stochastic processes. These processes satisfy a generalization of the notion of an increasing failure rate, and from a physical point of view seem very suitable for describing the process of cumulative wear or damage. Our main interest with this model of wear will be an investigation of the random first time until breakdown. Since we

shall not assume that the wear process is known exactly, our results deal in the main with properties of the failure rate of the time until breakdown and with bounds on the moments of this random variable.

We shall now summarize partially the content and organization of this paper. In Section O, we state the definitions of certain fundamental mathematical concepts we will need such as total positivity of order two, monotone failure rate and Pólya frequency density of order two. These definitions will be referred to throughout the paper. In Section 1, we characterize the class of processes which we shall be concerned with throughout this paper. More exactly we shall consider those non-decreasing stochastic processes $\{Z_t \mid t > 0\}$ such that

$$P(Z_{t+h} > x \mid Z_t \leq x)$$

be non-increasing in x, for all $t > 0$, and $h > 0$ whenever defined. Such processes we shall call <u>stochastic wear processes.</u> We shall show that the compound-renewal process under certain non-parametric assumptions, as well as the translated Poisson process and the gamma process, are examples of stochastic wear processes.

In Section 2, we apply the concept of a stochastic wear process to an investigation of the failure rate of the random first time for the process to cross a random threshold. The key result here is that for stochastic wear processes with stationary, independent increments, the first passage time random variable has an increasing failure rate, regardless of the distribution of the random threshold. To show that this is not true in general, we provide a counter-example where the first passage time has first a decreasing failure rate, and then a constant failure rate for two successive intervals of known threshold levels. We also remark in this section that the assumption that $\{Z_t \mid t > 0\}$ is a stochastic wear process is equivalent to imposing a structuring or ordering of the failure rate function of the first passage time across a known threshold. By the above, we mean if we increase the known threshold level, then the failure rate function of the first passage time uniformly decreases.

In Section 3, we consider the first passage problem across a random threshold for the special important case of the compound renewal process. Assuming that the blows have an increasing failure rate, we obtain tight upper and lower bounds on the mean and variance of this first passage time. We also obtain for any non-decreasing stochastic process the Laplace-transform of this first passage time across a random threshold.

In Section 4, we investigate the operations on stochastic processes under which the structuring of the failure rate mentioned above is preserved. We find that this structuring is preserved under formation of a "k out of n" system and, with certain conditions, under convolution. We also investigate a strengthening of the concept of a stochastic wear process and, as an application, consider the case where the threshold is known probabilistically.

0. Preliminaries

The following definitions will be referred to frequently in this investigation of stochastic wear processes.

Definition 1:

A function $g(x,y) \geq 0$ of two variables ranging over linearly-ordered one dimensional sets X and Y respectively, is said to be totally positive of order 2 (TP_2) if for all $x_1 \leq x_2$, and

$y_1 \leq y_2$, then

$$\begin{vmatrix} g(x_1,y_1) & g(x_1,y_2) \\ g(x_2,y_1) & g(x_2,y_2) \end{vmatrix} \geq 0 \ .$$

<u>Definition 2:</u>

Let X be a random variable. Then X is said to have an increasing (decreasing) failure rate, IFR(DFR), if and only if $P(X \leq x + h \mid X > x)$ is non-decreasing (non-increasing) in x where defined and for all $h > 0$.

<u>Definition 3:</u>

Let X be a random variable. Then X is said to possess a Pólya frequency density of order 2 (PF_2) if and only if $\frac{f(x)}{F(x+\Delta)-F(x)}$ is non-decreasing in x , for all real Δ where defined and where $F(x) = P(X \leq x)$; $f(x) = \frac{d}{dx} F(x)$. The concepts of total positivity, monotone failure rate, and Pólya-frequency density are related in the following way:

<u>Lemma 0</u>

Let $\bar{F}(x) = P(X > x)$ and $f(x) = \frac{-d}{dx} \bar{F}(x)$.

Then

i) The random variable X is IFR if and only if $\bar{F}(x-y)$ is TP_2 in x and y .

ii) The random variable X is DFR if and only if $\bar{F}(x+y)$ is TP_2 in x and y where $x+y \geq 0$.

iii) The density function of the random variable X is a PF_2 density function if and only if $f(x-y)$ is TP_2 in x and y .

iv) If the random variable X possesses a PF_2 density, then X is an IFR random variable. The converse is not true. (See Barlow, Proschan, 1965)

1. <u>A Generalization of the Failure Rate Concept to Stochastic Processes</u>

Let $\{Z_t \mid t > 0\}$ be a non-decreasing stochastic process where the values at t, denoted Z_t, are finite, measurable functions on a probability space (Ω, A, P) to the real line. Let $Z_0 \equiv 0$ and the process be separable whenever necessary.

If Z_t is thought of as the cumulative wear on some component at time t, and failure occurs whenever the process exceeds x, then

(1) $$P(Z_{t+h} > x \mid Z_t \leq x)$$

is a natural generalization for stochastic processes of the failure rate (see definition 2). This is the probability that given that the component has not worn out at time t, then h units of time later, it will have worn out. We consider the assumption that (1) is non-increasing in x, whenever defined, for all $h > 0$, and $t > 0$. Perhaps a more intuitive way of viewing condition (1) is to require

(2) $$P(Z_{t+h} \leq x \mid Z_t \leq x)$$

to be non-decreasing in x.

<u>Definition 4</u>:

A non-decreasing stochastic process $\{Z_t \mid t > 0\}$ will be called a <u>stochastic wear process</u> if (1) is non-increasing in x, where defined, for all $h > 0$, and $t > 0$.

Lemma 1

The stochastic process $\{Z_t \mid t > 0\}$ is a stochastic wear process if and only if $P(Z_t \leq x)$ is TP_2 in x and t.

Proof: Write (1) as

$$\frac{F(x,t) - F(x,t+h)}{F(x,t)} \tag{3}$$

where $F(x,t) = P(Z_t \leq x)$. By noting that (1) or equivalently (3) is non-increasing in x if and only if $\frac{F(x,t+h)}{F(x,t)}$ is non-decreasing in x, we see $F(x,t)$ must be TP_2 in x and t.

Remark: Suppose $f(x,t) = \frac{\partial}{\partial x} F(x,t)$ exists. Then it can be shown as an easy consequence of the fact that $F(x,t)$ is TP_2 that

$$\frac{f(x,t)}{F(x,t)} \tag{4}$$

is non-decreasing in t, for all x. Now $\frac{f(x,t)}{F(x,t)}\,dx$ has the probabilistic interpretation that $P(Z_t \in (x-dx,x) \mid Z_t \leq x)$ is non-decreasing in t, again quite intuitive for non-decreasing processes. As examples of processes which are stochastic wear processes (according to Definition 4) we offer the following:

Example 1: Let Z_t be the compound renewal process, that is,

$$Z_t = \begin{cases} \sum_{i=1}^{N_s(t)} Y_i & \text{if } N_s(t) > 0 \\ 0 & \text{otherwise} \end{cases}$$

where $N_s(t)$ denotes the number of renewals in $(0,t]$ with respect to a spacing renewal process $\{S_k\}_{k=1}^{\infty}$ and $\{Y_i\}_{i=1}^{\infty}$ are

independent, identically distributed samples from a non-negative random variable Y, independent from the spacing random variable S.
This model of wear is appropriate if a component is subject to wear produced by a series of blows, the blows occurring according to a renewal process.

Lemma 2

If both the random variables S and Y possess PF_2 densities, then the compound renewal process is a stochastic wear process according to Definition 4.

Proof: By Lemma 1, it is equivalent to prove $F(x,t) = P(Z_t \leq x)$ is TP_2 in x and t. It can also be shown easily that if $f(x,t) = \frac{\partial}{\partial x} F(x,t)$ is TP_2 in x and t, then $F(x,t)$ is TP_2 in x and t. Hence, it suffices to show $f(x,t)$ is TP_2 in x and t.

Now $f(x,t) = \sum_{n=0}^{\infty} P(N_s(t) = n)\, f_Y^{n*}(x)$ where $*$ represents convolution. However, since both S and Y have PF_2 densities, it follows that $P(N_s(t) = n)$ is TP_2 in n and t and $f_Y^{n*}(x)$ is TP_2 in n and x (Karlin, Proschan 1960). Hence, since total positivity is preserved under convolution (Pólya-Szegö, 1925), we have $f(x,t)$ is TP_2 in x and t.

Example 2: Let $Z_t = N_s(t)$ where $N_s(t)$ denotes the number of renewals in $(0,t]$ where S, the spacing random variable, has an increasing failure rate. This is a weaker condition than requiring S to have a PF_2 density. If $[x]$ denotes the greatest integer contained in x, then

$$F(x,t) = 1-F_S^{([x]+1)*}(t)$$

is TP_2 in x and t if $F_S(x) = P(S \leq X)$ is IFR (Barlow, Proschan, 1965). Hence $\{Z_t \mid t > 0\}$ is a stochastic wear process.

Remark: $F(x,t)$ is also TP_2 in translation of t, for each fixed x since the IFR property is preserved under convolutions (Barlow and Proschan, 1965).

Example 3: Let $Z_t = Y_t + \alpha t$, where c and α are constants greater than zero, and Y_t the compound Poisson with constant blows, i.e.,

$$Y_t = \begin{cases} c \sum_{i=1}^{N(t)} 1 & \text{if } N(t) > 0 \\ 0 & \text{elsewhere} \end{cases}$$

where $P(N(t) = n) = \dfrac{e^{-\lambda t}(\lambda t)^n}{n!}$ $n = 0,1,2...$

This model of wear is appropriate for a component which, in addition to undergoing wear due to a series of blows, is undergoing an abrasive continuous wear. We shall show in Section 4 as a consequence of Theorem 4 that Z_t is a stochastic wear process.

Remark: In this case, the random variable Z_t has a discrete PF_2 density for each t and hence possesses an increasing failure rate for each t.

Example 4: Let $\{Z_t \mid t > 0\}$ be the gamma process, that is,

$$f(x,t) = \begin{cases} \dfrac{e^{-x}x^{t-1}}{\Gamma(t)} & x \geq 0, \quad t > 0 \\ 0 & \text{otherwise.} \end{cases}$$

In the manner of Smith (1958), one can consider Z_t as the limiting form of a compound renewal process where the blows, in general, are very small in magnitude, but occur more and more frequently. Since it is well known that $f(x,t)$ is TP_2 in x and t, then we see that Z_t is a stochastic wear process. However Z_t is not IFR for every t. In fact, if $t < 1$, Z_t is a DFR random variable; if $t > 1$, Z_t is an IFR random variable, and Z_1 is both IFR and DFR since the failure rate is constant.

Not all non-decreasing stochastic processes are stochastic wear processes, i.e., it is not true in general that is, $P(Z_{t+h} > x \mid Z_t \leq x)$ is non-increasing in x, for consider the process $\{Z_t \mid t > 0\}$ where

$$Z_t = \begin{cases} Y_t & \text{if} \quad t \leq 1 \\ Y_1 & \text{if} \quad t > 1 \text{ and } Y_1 \leq 1 \text{ or } Y_1 > 2 \\ Y_{1+3} & \text{if} \quad t > 1 \text{ and } 1 < Y_1 \leq 2 \end{cases}$$

where $\{Y_t \mid t > 0\}$ is the gamma process. Then it can be shown that

$$P(Z_2 > 2 \mid Z_1 \leq 2) > P(Z_2 > 1 \mid Z_1 \leq 1) \text{ and hence}$$

$\{Z_t \mid t > 0\}$ is not a stochastic wear process.

2. Failure Rate of First Passage Random Variable

Let $$U(x) = \begin{cases} 1 & \text{if } x \geq 0 \\ 0 & \text{otherwise} \end{cases} .$$

Then, since $U(x)$ is Borel-measureable, the process $\{U(x-Z_t) \mid t > 0\}$ will be a measurable process for any fixed x (Halmos 1950). Then, by Fubini's Theorem, and in the manner of Smith (1958) we can introduce the random variable

$$T_x = \int_0^\infty U(x - Z_t)dt$$

Clearly, T_x is the first passage time of the non-decreasing process $\{Z_t \mid t > 0\}$ across the barrier or threshold at a known height of x.

We shall now investigate the properties of the failure rate of T_x, under the assumption that $\{Z_t \mid t > 0\}$ be a stochastic wear process.

Theorem 1. Let $\{Z_t \mid t > 0\}$ be a stochastic wear process with stationary, independent increments. Then for every $x > 0$, T_x is an IFR random variable.

Proof: We use the following lemma, a proof of which can be found in Barlow, Proschan (1965).

Lemma 3

If $F_1(0) = F_2(0)$ and $F_1(y) \geq F_2(y)$ for all $0 \leq y \leq x$ and if $Q(y) \geq 0$ is non-increasing on $[0,x]$, then

$$\int_0^x Q(y)\, dF_1(y) \geq \int_0^x Q(y)\, dF_2(y)$$

whenever the integrals exist.

Proof of Theorem 1: We must show for every $x > 0$ and $h > 0$, that

$$(5) \qquad P(T_x \leq t + h \mid T_x > t)$$

is non-decreasing in t where defined. Now, since

$$P(T_x \leq t) = P(Z_t > x)$$

then (5) can be expressed as

$$(6) \qquad \frac{\int_0^x [1-F(x-y,h)]d_y F(y,t)}{F(x,t)}$$

where $F(x,t) = P(Z_t \leq x)$.

Now (6) can be rewritten as

$$(7) \qquad 1 - \int_0^x F(x-y,h)d_y\left(\frac{F(y,t)}{F(x,t)}\right)$$

By Lemma 1, $F(x,t)$ is TP_2 in x and t . Hence

$$\frac{F(y,t_1)}{F(x,t_1)} \geq \frac{F(y,t_2)}{F(x,t_2)} \qquad \text{whenever } t_1 \leq t_2 \text{ and } 0 \leq y \leq x .$$

Also, we note $F(x-y,h) \geq 0$ is non-increasing in y on $[0,x]$. Hence, applying Lemma 3, we obtain

$$(8) \qquad \int_0^x F(x-y,h)d_y\left(\frac{F(y,t_1)}{F(x,t_1)}\right) \geq \int_0^x F(x-y,h)d_y\left(\frac{F(y,t_2)}{F(x,t_2)}\right) .$$

Hence, by (7) and (8), we obtain (5) is non-decreasing in t , for all $x > 0$.

Corollary

Let $\{Z_t \mid t > 0\}$ be a stochastic wear process with stationary, independent increments. Let X be an arbitrary non-negative random variable. Let T_X , a random variable, denote the first time the process exceeds a barrier at a height of X . Then T_X is an IFR random variable.

Proof: The proof follows trivially since T_x is IFR for all $x > 0$.

Remark concerning Theorem 1: The fact that the random variable T_X has an increasing failure rate implies that the random variable T_X

has the following, useful, inherited properties (for a proof of these properties and others, see Barlow and Proschan, 1965).

i) $E(T_X^r) \leq \Gamma(r+1)\ [E(T_X)]^r$ if $r \geq 1$

$E(T_X^r) \geq \Gamma(r+1)\ [E(T_X)]^r$ if $0 < r < 1$

where $\Gamma(r) = \int_0^\infty u^{r-1} e^{-u} du$

Note:

Taking $r = 2$ we see that (i) implies that the random variable T_X has a coefficient of variation less than or equal to 1.

ii) $P(T_X \geq ET_X) = P(Z_{ET_X} \leq X) \geq e^{-1}$

that is, the probability that at time ET_X , the process has not yet crossed the barrier at a height of X , is greater than or equal to e^{-1} .

iii) If components are replaced upon failure where T_X is the life of the component, then

$$\text{Var } N(t) \leq EN(t) \leq t/ET_X$$

and $P(N(t) \leq n) \geq \sum_{j=0}^{n} \left[(t/ET_X)^j \ \exp\ (-t/ET_X)/j! \right]$ where $N(t)$ is the number of spares needed to last a length of time t .

A natural question at this point is, "Will T_X have an increasing failure rate, in general, if $\{Z_t \mid t > 0\}$ is a non-decreasing process?" The answer is no as the following counter-example demonstrates.

Counter-Example: Let $Z_t = N_S(t)$ denote the number of renewals in $(0,t]$ where S , the spacing random variable, is the gamma random variable with parameter $\alpha < 1$; that is, let

$$f_s(x) = \begin{cases} \dfrac{\lambda^\alpha x^{\alpha-1} e^{-\lambda x}}{\Gamma(\alpha)} & \text{if } x \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

where $f_s(x)$ is the density functions of the random variable S.

Then, since

$$P(T_x \leq t) = F_s^{([x]+1)*}(t), \quad x > 0$$

it can quickly be shown that

i) T_x has a decreasing failure rate if $([x]+1)\alpha < 1$

ii) T_x has a constant failure rate if $([x]+1)\alpha = 1$

iii) T_x has an increasing failure rate if $([x]+1)\alpha > 1$.

Hence we have an example where the random variable T_x does not have an increasing failure rate function for every value of the threshold index x.

<u>Lemma 4</u>

Let $\{Z_t \mid t > 0\}$ have stationary, independent, non-negative increments. Then

$$ET_x \geq \frac{x}{EZ_1} \qquad \text{for all} \quad x > 0 .$$

The inequality is sharp.

Proof:

$$ET_x = E\{\int_0^\infty U(x-Z_t)dt\} = \int_0^\infty E(U(x-Z_t))dt = \int_0^\infty F(x,t)dt$$

Now, since $\{Z_t \mid t > 0\}$ is a non-decreasing porcess, we can say

$$ET_x = \int_0^\infty F(x,t)dt \geq h \sum_{n=1}^{\infty} F(x,n,h) \quad \text{for all} \quad h > 0 .$$

But, since $\{Z_t \mid t > 0\}$ has stationary, independent increments,

$$F(x,n,h) = F(x,h)^{n*}$$

Hence $\sum_{n=1}^{\infty} F(x,nh) = EN_{Z_h}(x)$

where $N_{Z_h}(x)$ denotes the number of renewals in $(0,x]$ where the spacing random variable is Z_h. Hence, we obtain

$$ET_x = \int_0^\infty F(x,t)dt \geq h\, EN_{Z_h}(x) \geq h \frac{x}{EZ_h} -1$$

and since $EZ_h = hEZ$, we obtain the desired result by letting h go to zero.

The lower bound is attained by the trivial process $\{Z_t = t\, EZ_1 \mid t > 0\}$. ||

<u>Corollary</u>

There is no non-trivial stochastic process $\{Z_t \mid t > 0\}$ with stationary, independent, non-negative increments for which Z_t, a non-discrete random variable, is IFR for each t.

Proof:

$$ET_x = \int_0^\infty F(x,t)dt \quad h + h \sum_{n=1}^{\infty} F(x,nh) .$$

Hence, $ET_x \leq h + h\, EN_{Z_h}(x)$ for all $h > 0$. Now, if Z_h were IFR for all $h > 0$, then by Barlow, Proschan (1965), we would have

$$EN_{Z_h}(x) \leq \frac{x}{EZ_h} = \frac{x}{hEZ_1} \qquad \text{for all} \quad h > 0 .$$

Hence, letting h go to zero, we get

$$ET_x \leq \frac{x}{EZ_1}$$

But from Lemma 4, we have

$$ET_x \geq \frac{x}{EZ_1} \quad .$$

Hence

$$ET_x = \int_0^\infty F(x,t)dt = \frac{x}{EZ_1}$$

Hence, taking Laplace transforms on both sides with respect to x , we obtain

$$\frac{-1}{s \log E(e^{-sZ_1})} = \frac{1}{s^2 EZ_1}$$

or equivalently $E(e^{-sZ_1}) = e^{-sEZ_1}$ i.e., $\{Z_t | t > 0\}$ is the trivial degenerate process.

Remark: Smith (1958) in his investigations of arbitrary stochastic processes with stationary, independent increments demonstrates that

$$ET_x = \frac{x}{EZ_1} + O(x)$$

and $ET_{x+h} - ET_x = \frac{h}{EZ_1} + O(x)$ for all $h > 0$.

Hence, even though $\{Z_t \mid t > 0\}$ is an unknown stochastic wear process with stationary, independent increments, many of the properties of the random variable T_x are known.

<u>Lemma 5</u>

Let $\{Z_t \mid t > 0\}$ be an arbitrary stochastic wear process. Then, for every $L > 0$, the ratio of the mean time the process spends under x in the interval $(0,L)$ to the mean time the process spends under x is the interval $(0, \infty)$ is non-increasing in x .

Proof: By Lemma 1, we have

$$(9) \qquad \begin{vmatrix} F(x_1,u) & F(x_1,v) \\ F(x_2,u) & F(x_2,v) \end{vmatrix} \geq 0$$

whenever $x_1 \leq x_2$, $u \leq v$.

Integrating (9) on u from 0 to L , and on v from L to ∞ , we obtain

$$(10) \qquad \begin{vmatrix} \int_0^L F(x_1,u)du & \int_L^\infty F(x_1,v)dv \\ \int_0^L F(x_2,u)du & \int_L^\infty F(x_2,v)dv \end{vmatrix} \geq 0 \ .$$

Adding column one to column two of (10), and noting that

$$E\left(\int_0^L U(x-Z_t)dt\right) = \int_0^L E(U(x-Z_t))dt = \int_0^L F(x,t)dt$$

is the mean time the process spends under x in the interval $(0,L)$, we obtain the desired result.

<u>Definition 5</u>:

A process $\{Y_t \mid t > 0\}$ will be said to possess the structured failure rate property if

$$(11) \qquad P(Y_t \leq y + h \mid Y_t > y)$$

is non-increasing in t , whenever (11) is defined, for all y and $h > 0$.

Lemma 6

The stochastic process $\{Z_t \mid t > 0\}$ is a stochastic wear process if and only if the process $\{T_x \mid x > 0\}$ possesses the structured failure rate property. Proof is obvious.

Remark: Lemma 6 agrees with our intuition since we would expect that as the threshold level x increases, the failure rate function of the random variable T_x should decrease. We also observe that under the conditions of Theorem 1, that the failure rate function of the random variable T_x, that is,

$$P(T_x \leq t + h \mid T_x > t) \tag{12}$$

is not only monotone in t, but also monotone in x.

We shall have more to say about this structuring of the failure rate property in Section 4. Now we shall turn our attention towards the important compound-renewal wear process.

3. The First Passage Time Over A Random Threshold for the Compound Renewal Process

Let

$$Z_t = \begin{cases} \sum_{i=1}^{N_s(t)} Y_i & \text{if } N_s(t) > 0 \\ 0 & \text{otherwise} \end{cases}$$

where $N_s(t)$ denotes the number of renewals in $(0,t]$ with respect to a spacing renewal process $\{S_k\}_{k=1}^{\infty}$. Let $\{Y_i\}_{i=1}^{\infty}$ be independent, identically distributed observations of a non-negative random variable Y, independent from the spacing random variable S.

As noted before, the compound renewal process is appropriate whenever a component undergoes wear produced by a series of blows, the blows occurring according to a renewal process. In sections one and two, we have seen under certain non-parametric assumptions concerning the spacings between blows and the size of the blows, that the first passage time over a random threshold has an increasing failure rate. This is true regardless of how the random threshold values are distributed.

Now we shall obtain tight bounds on the mean and variance for the first time for a compound renewal process to cross the random threshold. We shall assume that only the mean of the random threshold is known, and that the random magnitude of the blows has a non-decreasing failure rate, i.e., $P(Y \leq y+h \mid Y > y)$ is non-decreasing in y whenever defined. This includes the gamma and Weibull distributions for some range of the parameters involved, as well as the exponential and degenerate distributions. The following lemma is pertinent to this section.

<u>Lemma 7</u>

Let $Z_t \mid t > 0$ be an arbitrary non-decreasing process where $Z_0 \equiv 0$. If T_X is the first time for the process to cross a random threshold X, then

$$(13) \qquad E(e^{-sT_X}) = P(Z_V > X)$$

where V is the exponential random variable with mean $\frac{1}{s}$.

Proof:

$$(14) \qquad E(e^{-sT_X}) = -\int_0^\infty e^{-st} dG(t)$$

where $G(t) = P(T_X > t)$.

Integrating (14) by parts, we obtain

$$(15) \qquad E(e^{-sT_X}) = 1 - s\int_0^\infty e^{-st}G(t)dt$$

Now $G(t) = \int_0^\infty P(T_x > t)dF(x)$ where $F(x) = P(X \leq x)$. Hence,

$$(16) \qquad G(t) = \int_0^\infty P(Z_t \leq x)dF(x) = P(Z_t \leq X)$$

Hence, by (15) and (16), we obtain (13).

Corollary

$$\int_0^\infty e^{-qx}E(e^{-sT_x})dx = \frac{1}{q} - \frac{s}{q}\int_0^\infty e^{-su}E(e^{-qZ_u})du$$

Proof: In Lemma 7, let X denote the exponential random variable with mean $\frac{1}{q}$. Then from (13), we have

$$(17) \qquad E(e^{-sT_X}) = \int_0^\infty qe^{-qx}E(e^{-sT_x})dx$$

$$= 1 - \int_0^\infty\int_0^\infty P(Z_u \quad x)se^{-su}qe^{-qx}dx\, du$$

$$(18) \qquad = 1 - \int_0^\infty E(e^{-qZ_u})se^{-su}du .$$

Hence, dividing (18) by q , we have the desired result.

Corollary

For the case of the compound-renewal process defined in Section 3, we obtain

$$(19) \qquad \int_0^\infty e^{-qx}E(e^{-pT_x})dx = \frac{\tilde{f}_s(p)(1-\tilde{f}(q))}{q(1-\tilde{f}_Y(q)\tilde{f}_s(P))}$$

where $\tilde{f}_Y(q) = E(e^{-qY})$; $\tilde{f}_s(p) = E(e^{-pS})$.

Proof: Since the double Laplace transform of Z_t for the compound-renewal process is well known, the result follows easily.

Formula (19) is a very attractive general formula, but unfortunately does not yield explicit solutions except to rather simple problems. Alternatively we will calculate the first and second moments of T_x directly. For these calculations, the following lemma is required.

<u>Lemma 8</u>

Let $N_Y(t)$ denote the number of renewals in $(0,t]$ where the spacing random variable Y is distributed according to $F_Y(x)$.

If $$H_Y^{(\ell)}(t) = \sum_{n=1}^{\infty} n^{\ell} F_Y^{n*}(t) \qquad \ell = 1,2,\ldots$$

then $$E(N_Y(t)^n) = \sum_{j=1}^{n} \binom{n}{j}(-1)^{j+1} H_Y^{(n-j)}(t) \qquad n = 1,2,\ldots$$

Proof:

(21) $$E(N_Y(t)^n) = \sum_{k=0}^{\infty} k^n P(N_Y(t) = k)$$
$$= \sum_{k=0}^{\infty} k^n (F_Y^{k*}(t) - F_Y^{(k+1)*}(t))$$
$$= \sum_{k=1}^{\infty} (k^n - (k-1)^n)\, F_Y^{k*}(t) \quad .$$

But

(22) $$k^n - (k-1)^n = \sum_{j=1}^{n} \binom{n}{j}(-1)^{j+1} k^{n-j} \quad .$$

Hence, (21) and (22) yield

$$E(N_Y(t)^n) = \sum_{k=1}^{\infty} \sum_{j=1}^{n} \binom{n}{j} (-1)^{n+1} k^{n-j} F_Y^{k*}(t)$$

$$= \sum_{j=1}^{n} \binom{n}{j} (-1)^{j+1} H_Y^{(n-j)}(t)$$

Note: it can also be shown

$$H_Y^{(\ell)}(t) = E(N_Y(t)^{\ell}) + \int_0^t F_Y(t-u)dH_Y^{(\ell)}(u) \qquad \ell = 1,2,\ldots \quad .$$

Also, we see $H^{(0)}(t) = E\, N_Y(t)$.

Theorem 2. Let $\{Z_t \mid t > 0\}$ be the compound-renewal process defined in Section 3. Let T_X be the first time the process exceeds the random threshold X . If Y , the random magnitude of the blows, has a non-decreasing failure rate, then

$$\frac{\mu_s \mu}{\mu_Y} \leq ET_X \leq \frac{\mu_s \mu}{\mu_Y} + \mu_s$$

$$\frac{\sigma_s^2 \mu}{\mu_Y} \leq \mathrm{Var}\, T_X \leq \sigma_s^2 + \frac{(\sigma_s^2 + \mu_s^2)}{\mu_Y}$$

where

$\mu_s = E(S)$ and $\sigma_s^2 = \mathrm{Var}\,(S)$ where S is the random spacings between blows.

$\mu_Y = E(Y)$ where Y is the random magnitude of the blows and

$\mu = E(X)$ where X is the random height of the barrier.

The above upper bounds are tight.

Proof:

$$E(T_X \mid X = x) = \int_0^\infty F(x,t)dt$$

Now since $F(x,t) = \sum_{n=0}^{\infty} P(N_s(t) = n) F_Y^{n*}(x)$ where $F_Y^{0*}(x) \equiv 1$, we have

$$E(T_X \mid X = x) = \sum_{n=0}^{\infty} F_Y^{n*}(x) \int_0^\infty P(N_s(t) = n)dt$$

But

$$\int_0^\infty P(N_s(t) = n)dt = \int_0^\infty (\bar{F}_s^{(n+1)*}(t) - \bar{F}_s^{n*}(t))dt$$

$$= (n+1)\mu_s - n\mu_s = \mu_s .$$

Hence

(23) $$E(T_X \mid X = x) = \mu_s(1 + EN_Y(x))$$

since

$$EN_Y(x) = \sum_{n=1}^{\infty} F_Y^{n*}(x) .$$

Similarily, we obtain $E(T_X^2 \mid X = x)$ by noting

$$E(T_X^2 \mid X = x) = 2\int_0^\infty t\, F(x,t)dt$$

$$= 2 \sum_{n=0}^{\infty} F_Y^{n*}(x) \int_0^\infty t(\bar{F}_s^{(n+1)*}(t) - \bar{F}_s^{n*}(t))dt$$

$$= \sum_{n=0}^{\infty} F_Y^{n*}(x) [\sigma_s^2 + 1 + 2n\, \mu_s^2]$$

Hence

(24) $$E(T_X^2 \mid X = x) = 2\, \mu_s^2\, H_Y^{(1)} + (\sigma_s^2 + \mu_s^2)\, H_Y^{(0)}(x)$$

By Lemma 8

(25) $$2H_Y^{(1)}(x) = E(N_Y^2(x)) + H_Y^{(0)}(x)$$

Hence, substituting (25) into (24), together with (23), we obtain

$$\text{(26)} \qquad \mathrm{Var}(T_X \mid X = x) = \mu_s^2(\mathrm{Var}\ N_Y(x)) + \sigma_s^2(EN_Y(x) + 1)$$

Now, as shown by Barlow and Proschan (1965), if Y has an increasing failure rate, then

$$\text{(27)} \qquad \mathrm{Var}\ N_Y(x) \leq EN_Y(x) \leq \frac{x}{\mu_Y} \quad .$$

Hence, using (23), (26), and (27), and unconditioning on X, we obtain the upper bounds in Theorem 2. To obtain the lower bounds, we use the fact that

$$\mathrm{Var}\ N_Y(x) \geq 0 \ ; \ EN_Y(x) \geq \frac{x}{\mu_Y} - 1$$

and uncondition on X .

The upper bound of Theorem 2 is attained when Y is the exponential random variable.

Remark: Smith (1955) has shown if $\{Z_t \mid t > 0\}$ is the compound renewal process, then Z_t is asymptotically normal with mean tEZ_1 , and Variance $t\ \mathrm{Var}\ Z_1$, where $EZ_1 = \frac{\mu_Y}{\mu_s}$, and Variance $Z_1 = \frac{\sigma_Y^2}{\mu_s} + \frac{Y^2 \sigma_s^2}{\mu_s^3}$. Also, it is clear that if a large number of steps are necessary to reach the barrier, then the distribution of T_x will be nearly normal. In fact, since

$$P(T_x < t) = P(Z_t > x) \ ,$$

it can be shown that T_x is asymptotically normal with mean $\frac{x}{EZ_1}$ and Variance $\frac{x\ \mathrm{Var}\ Z_1}{(EZ_1)^3}$, in complete analogy with the asymptotic distribution of the renewal random variable $N(x)$, for renewal process (Feller, (1957)).

4. Preservation of Structural Failure Rate Property and An Extension

We noted in Lemma 6, that the assumption that $\{Z_t \mid t > 0\}$ be a stochastic wear process was equivalent to the process $\{T_x \mid x > 0\}$ possessing a structuring of its failure rate as defined by Definition 5.

It is natural to inquire under what operations with stochastic processes this structuring of the failure rates is preserved. This is important in investigating structures such as parallel and series systems where the wear or damage to a structure is actually the minimum or the maximum of a sequence of different wear processes. Also, since the total wear on a structure is often composed of two distinct types of wear which add together, it is important to investigate conditions for which this structuring is preserved under convolution.

Theorem 3. Let $\{Y_t^{(i)} \mid t > 0\}$, $i = 1,2,\ldots,n$ be a sequence of stochastic processes such that for each t , the set of random variables $\{Y_t^{(i)}\}$ $i = 1,2,\ldots,n$ are independent and identically distributed. Define $\{M_t^{(k)} \mid t > 0\}$, $k = 1,2,\ldots,n$ as a sequence of stochastic processes where the random variable $M_t^{(k)}$ denotes the k^{th} largest over $i = 1,2,\ldots,n$ of the set $\{Y_t^{(i)}\}$. If each process $\{Y_t^{(i)} \mid t > 0\}$ $i = 1,2,\ldots,n$ possesses the structured failure rate property of Definition 5, then the stochastic processes $\{M_t^{(k)} \mid t > 0\}$, $k = 1,2,\ldots,n$, do likewise.

Proof: We shall need the following lemma.

Lemma 9

If $\{Y_t \mid t > 0\}$ possesses the structured failure rate property of Definition 5, then

$P(Y_t \leq x)$

is non-increasing in t for all x, and is TP_2 in x and t.

Proof of Lemma 9: Let $F(x,t) = P(Y_t \leq x)$; $f(x,t) = \frac{\partial}{\partial_x} F(x,t)$.
Then directly from Definition 5, we have $1 - F(x,t)$ is TP_2 in x and t since

$$P(Y_t \leq x + h) \mid Y_t > x) = \frac{F(x+h,\ t) - F(x,t)}{1 - F(x,t)}$$

As an easy consequence, we get $\frac{f(x,t)}{1-F(x,t)}$ is non-increasing in t.

Hence, since $1-F(x,t) = \exp(- \int_{-\infty}^{x} (\frac{f(u,t)}{1-F(u,t)})du)$ we have that $F(x,t)$ is non-increasing in t.

Proof of Theorem 3. Let

$$P(Y_t^{(i)} > x) = \bar{G}(x,t) \ ; \ - \frac{d}{dx} \bar{G}(x,t) = g(x,t) \qquad i = 1,2,\ldots,n$$

$$P(M_t^{(k)} > x) = \bar{F}_k(x,t); \ - \frac{d}{dx} \bar{F}_k(x,t) = f_k(x,t) \quad k = 1,2,\ldots,n$$

Then,

$$\bar{F}_k(x,t) = \sum_{i=k}^{n} \binom{n}{i} (\bar{G}(x,t))^i (G(x,t))^{n-i} .$$

Now, in the manner of Barlow, Proschan (1965), we have

$$\bar{F}_k(x,t) = \left(\frac{\Gamma(n+1)}{\Gamma(k)\ (n+1-k)}\right)\left(\int_0^{\bar{G}(x,t)} u^{k-1}(1-u)^{n-k}\ du\right)$$

Hence, we obtain

$$\frac{\bar{F}_k(x,t)}{f_k(x,t)} = \frac{\bar{G}(x,t)}{g(x,t)} \int_0^1 v^{k-1} \left[\frac{(1-v)}{G(x,t)} + v\right]^{n-k} dv$$

Now, since $\{Y_t^{(i)} \mid t > 0\}$ has the structured failure rate property, it is easily seen $\frac{\bar{G}(x,t)}{g(x,t)}$ is non-decreasing in t. Hence, using Lemma 9, we obtain $\frac{\bar{F}_k(x,t)}{f_k(x,t)}$ non-decreasing in t, i.e., $\{M_t^{(k)} \mid t > 0\}$ possesses a structured failure rate $k = 1,2,\ldots,n$.

Remark: If $k=n$, we can weaken Theorem 3 by eliminating the condition that $\{Y_t^{(i)}\}$ $i = 1,2,\ldots,n$ be identically distributed r.v.'s for each t. To see this, note that if $k = n$, then

$$\bar{F}_n(x,t) = \prod_{i=1}^{n} \bar{G}_i(x,t) \quad \text{where} \quad \bar{G}_i(x,t) = P(Y_t^{(i)} > x) .$$

Hence, $\frac{d}{dx}\frac{d}{dt} \log \bar{F}_n(x,t) = \sum_{i=1}^{n} \frac{d}{dx}\frac{d}{dt} \log \bar{G}_i(x,t)$

Now, $p(x,y) > 0$ is TP_2 in x, y if and only if $\frac{d}{dx}\frac{d}{dy} \log p(x,y) \geq 0$ (Karlin 1957). Hence, $\bar{G}_i(x,g)$ TP_2 in x, t for each $i = 1,2,\ldots,n$ implies $\bar{F}_n(x,t)$ is TP_2 in x, t.

Theorem 4.

a) Let $\{Y_t^{(1)} \mid t > 0\}$ and $\{Y_t^{(2)} \mid t > 0\}$ be arbitrary stochastic process where $Y_t^{(1)}$ and $Y_t^{(2)}$ are independent random variables for each t.

b) Let $\{Y_t^{(1)} \mid t > 0\}$ possess the structured failure rate property of Definition 5 and each random variable $Y_t^{(1)}$, $t > 0$ possess an increasing failure rate.

c) Let $g_2(x,t)$ be TP_2 in x and t where $g_2(x,t)$ denotes the density function of the random variable $Y_t^{(2)}$. This, as can be easily shown, implies $\{Y_t^{(2)} \mid t > 0\}$ possesses a structured failure rate. Finally, assume that each random variable $Y_t^{(2)}$, $t > 0$ possesses a PF_2 density. Then under conditions (a), (b), (c), the stochastic process

$$\{Y_t^{(1)} + Y_t^{(2)} \mid t > 0\}$$

possesses the structured failure rate property of Definition 5 and for each t, the random variable $Y_t^{(1)} + Y_t^{(2)}$ if IFR.

Note:

We have seen from Theorem 1 and Lemma 5 that the process $\{T_x \mid x > 0\}$ satisfies condition (b) if the underlying wear process is a stochastic wear process with stationary, independent increments. Other examples of processes satisfying conditions (b), and (c) are the compound Poisson with constant blows, and the degenerate process. Before we can prove Theorem 4, we must state the following lemma.

Lemma 10

Let $g(x,y) = \int_{-\infty}^{\infty} f(x-u,y)\, h(u,y)du$

If both $f(x,y)$ and $h(x,y)$ are TP_2 in x and y and TP_2 in translations of x for each y, then $g(x,y)$ is TP_2 in x and y and TP_2 in translations of x for each y. Proof is due to Ghurye and Wallace (1959).

Proof of Theorem 4:

Let $\bar{H}(x,t) = P(Y_t^{(1)} + Y_t^{(2)} > x)$

and $G_i(x,t) = P(Y_t^{(i)} \leq x) \qquad i = 1,2$.

Then it is clear that

$$\bar{H}(x,t) = \int_{-\infty}^{\infty} (1 - G_1(x-y,t))d_y G_2(y,t)$$

From condition (b) and Lemma 9, we have $1-G_1(x,t)$ is TP_2 in x and t. From condition (b) and Lemma 0, we have $1-G_1(x-y,t)$ is TP_2 in x and y for each t. From (c) and Lemma 0 we have $d_y G_2(y,t)$ is TP_2 in y and t and TP_2 in translations of y for each t. Hence, by an application of Lemma 10, we have $\bar{H}(x,t)$

is TP_2 in x and t. Hence by Lemma 9,

$$\{Y_t^{(1)} + Y_t^{(2)} \mid t > 0\}$$

possesses the structured failure rate property. Also, since $\bar{H}(x,t)$ is TP_2 in translations of x for each t, then by Lemma 0, we obtain that for each $t > 0$, the random variable $Y_t^{(1)} + Y_t^{(2)}$ is IFR.

An extension of the stochastic wear process characterization of Definition 4 is to require $q(t,x)$, the density function of the random variable T_x, to be TP_2 in x and t. This indeed is a strengthening assumption since $q(t,x)$ TP_2 implies that $P(T_x > t)$ is TP_2 in x, t. This stronger condition is of interest in the problem of esti ating the threshold level, having observed the time to failure. We note that this stronger assumption is equivalent to the failure time statistic possessing a monotone likelihood ratio (for a definition and applications, see Karlin, Rubin 1956).

As examples of processes possessing this stronger property, consider the following:

Example 1: Compound Poisson, with PF_2 blows, i.e., let

$$F(x,t) = P(Z_t \leq t) = \sum_{n=0}^{\infty} \frac{e^{-\lambda t}(\lambda t)^n}{n!} F_Y^{n*}(x)$$

where Y has a PF_2 density. Then

$$P(t < T_x < t + dt) = q(t,x)dt = \lambda \int_0^x (1-F_Y(x,y))d_yF(y,t)dt$$

Now, by example 1 of section 1, $d_yF(y,t)$ is TP_2 in y, t. Also, it is well-known that $1-F_Y(x-y)$ is TP_2 in x, y if Y is PF_2.

Hence, by Pólya-Szegö (1925) $q(t,x)$ is TP_2 in t, x.

__Example 2__: Let $\{Z_t \mid t > 0\}$ be a temporarily homogeneous strong Markoff process with a non-negative drift. Let $Z_0 = 0$. Let the process possess a realization with almost every path function continuous, where continuity is with respect to a suitable order topology. If $q(t,x)dt = P(t < T_x < t + dt)$, then $q(t,x)$ is TP_2 in t and x (see Karlin, 1964). Examples include the Poisson, Wiener and degenerate processes.

As an application of the assumption that $q(t,x)$ is TP_2 in t and x, we consider the case when the threshold X is known probabilistically, i.e.,

$$P(X \leq x) = F(x) .$$

In addition, suppose we have a collection of replacement strategies, say $\{\phi_i(t)\}$ $i = 1,2,\ldots,n$, depending on the observed time or age at failure of the component. Such strategies might be to use a block replacement policy, age replacement policy, to replace more or less often, or perhaps not to replace at all. Let $L_i(x)$ be the loss incurred if strategy i is used and x is the true value of the threshold.

Consider

$$\lambda_{ij}(t) = \int_0^\infty (L_i(x) - L_j(x))q(t,x)dF(x) .$$

We notice if $\lambda_{ij}(t) < 0$, then action i is to be preferred over action j when t is observed; similarily if $\lambda_{ij}(t) > 0$, action j is preferred over action i.

Now if $L_i(x) - L_j(x)$ has at most one sign change, say from negative to positive, then by the variation diminishing property (Karlin (1964)) of totally positive function, there exists a t_0 such that

1) if $t \leq t_0$, action i is preferred over action j

2) if $t > t_0$, action j is preferred over action i .

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